The Expected Value of an Everywhere Stopped Martingale

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Abstract

If the coordinate random variables $\{X_t\}$ on either $C[0,\infty)$ or $D[0,\infty)$ form a martingale, then for every stopping time τ which is <u>everywhere</u> finite, $E(X_\tau)$, if defined, equals $E(X_0)$. This version of the optional sampling theorem is not covered by Doob's classical result.

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In this paper, Ω may be thought of as either $C[0,\infty)$, the space of real-valued continuous functions on $[0,\infty)$, or as $D[0,\infty)$, the space of real-valued, right continuous functions on $[0,\infty)$ which have finite left limits. It is well-known that Ω, equipped with a suitable metric, is a complete, separable metric space. (See section 1.3 of [3] and section 2 of [4] for example.) Let F be the Borel σ-field on Ω. For a nonnegative real number t, let X_t be the coordinate map on Ω defined by $X_t(\omega) = \omega(t)$, $\omega \in \Omega$, and let F_t be the σ-field generated by the collection of random variables $\{X_s, 0 \leq x \leq t\}$. Let T be the collection of F_t-adapted stopping times τ on Ω which are everywhere finite; i.e. all functions τ on Ω such that $0 \leq \tau(\omega) < \infty$ for all $\omega \in \Omega$ and $[\tau \leq t] \in F_t$ for all $t \geq 0$. Here is the main result of the paper.

<u>Theorem</u>. Let P be a probability measure on (Ω, F) under which $\{X_t\}$ is an F_t-adapted martingale. Then, for every $\tau \in T$, either

(a) $E^P(X_\tau) = E^P(X_0)$

or

(b) $E^P(X_\tau)$ is not defined, i.e. $E^P[\max(X_\tau,0)] = E^P[\max(-X_\tau,0)] = \infty$.

(E^P denotes the expected value under the probability measure P.)

The main technique used in the proof of this theorem is an adaptation to the continuous-time case of the stop rule induction method of Dubins and Savage [2]. Some additional notation is needed for the formulation.

Let Ω^* be the collection of all initial segments of paths in Ω; i.e. $p \in \Omega^*$

iff for some positive real number t and some $\omega \in \Omega$, p is the restriction of ω to $[0,t)$. By the <u>length</u> of an element in Ω^* we shall mean the length of its domain. For p_1, $p_2 \in \Omega^*$ with lengths t_1, t_2 respectively, p_1p_2 will stand for the function on $[0,t_1+t_2)$ defined by

$$\begin{aligned} p_1p_2(s) &= p_1(s), \quad && 0 \le s < t_1 \\ &= p_2(s-t_1), \quad && t_1 \le s < t_1+t_2. \end{aligned}$$

For $p \in \Omega^*$ of length t and $\omega \in \Omega$, $p\omega$ will stand for the function on $[0,\infty)$ defined by

$$\begin{aligned} p\omega(s) &= p(s), \quad && 0 \le s < t, \\ &= \omega(s-t), \quad && s \ge t. \end{aligned}$$

If Ω is $D[0,\infty)$, then $p_1p_2 \in \Omega^*$ and $p\omega \in \Omega$. However, this is not necessarily true when Ω is $C[0,\infty)$ because p_1p_2 may have a discontinuity at t_1 and $p\omega$ may have a discontinuity at t.

For $p \in \Omega^*$ of length t and $\tau \in T$, let $\tau[p]$ be the stopping time in T defined by

$$\begin{aligned} \tau[p](\omega) &= \tau(p\omega) - t \quad && \text{if } p\omega \in \Omega \text{ and } \tau(p\omega) \ge t, \\ &= 0 && \text{if } p\omega \in \Omega \text{ and } \tau(p\omega) < t, \\ &= 0 && \text{if } p\omega \notin \Omega. \end{aligned}$$

One can regard $\tau[p]$ as the additional time to wait given that the segment p has already occurred.

<u>Induction lemma</u>. Let $\phi(\tau)$ be a proposition for every $\tau \in T$. Assume

1. $\phi(\tau)$ holds if $\tau \equiv 0$,
2. $\phi(\tau)$ holds if $\phi(\tau[p])$ holds for every $p \in \Omega^*$ of length 1.

Then $\phi(\tau)$ holds for all $\tau \in T$.

<u>Proof</u>: Suppose there is a $\tau \in T$ for which $\phi(\tau)$ is false. By assumption 2 of the lemma, there exists a sequence $\{p_n\}$ of elements in Ω^* each of length 1, and a sequence $\{\tau_n\}$ of stopping times in T such that

a) $\tau_1 = \tau[p_1]$ and $\tau_{n+1} = \tau_n[p_{n+1}]$, $n \geq 1$,

b) $\phi(\tau_n)$ is false for all n.

Consider two cases. (The first case does not arise when Ω is $D[0,\infty)$.)

<u>Case i</u>) For some n, $p_1 \ldots p_n \notin \Omega^*$.

In this case, $p_1 \ldots p_n \omega \notin \Omega$ for any $\omega \in \Omega$. It is straightforward to check that $\tau_n = \tau[p_1 \ldots p_n]$ and, consequently, $\tau_n \equiv 0$. So, by b), we have a contradiction to assumption 1.

<u>Case ii</u>) For every n, $p_1 \ldots p_n \in \Omega^*$.

Let ω be the function on $[0,\infty)$ defined by

$$\omega(s) = p_n(s-n+1) \quad \text{if } n-1 \leq s < n$$

for n = 1, 2,... . Because Ω is either $C[0,\infty)$ or $D[0,\infty)$, the ω defined above for $n = 1, 2, \ldots$. Because Ω is either $C[0,\infty)$ or $D[0,\infty)$, the ω defined above belongs to Ω. Because τ is everywhere finite, $\tau(\omega) < \infty$. Let n be the positive integer such that $n-1 \le \tau(\omega) < n$. Plainly, $\tau_n \equiv 0$ and we get a contradiction in this case too.

The proof of the lemma is now complete.

Proof of the theorem: For $\tau \in \mathcal{T}$, let $\phi(\tau)$ be the proposition that, whenever P is a probability measure on $(\Omega,\mathcal{F})$ under which $\{X_t\}$ is an $\mathcal{F}_t$-adapted martingale, either (a) $E^P(X_\tau) = E^P(X_0)$ or (b) $E^P(X_\tau)$ is undefined.

The theorem will be proved once we verify the assumptions of the induction lemma.

Obviously, $\phi(\tau)$ holds if $\tau \equiv 0$. To verify assumption 2, suppose that $\tau \in \mathcal{T}$ is such that $\phi(\tau[p])$ holds for every p in Ω^* of length 1 and suppose P is a probability measure on $(\Omega,\mathcal{F})$ under which $\{X_t\}$ is an $\mathcal{F}_t$-adapted martingale.

Define $\tau' = \min(\tau,1)$ and let $\mathcal{F}' = \mathcal{F}_{\tau'}$ be the σ-field generated by the collection of random variables $\{X_{\min(\tau',s)}, s\ge 0\}$. From the right continuity of every ω, it is easy to see that $\mathcal{F}'$ is a countably generated sub σ-field of $\mathcal{F}$. Let $\{Q_\omega\}$ be a regular conditional probability distribution of P given $\mathcal{F}'$ which is proper in the sense that $Q_\omega(A) = 1_A(\omega)$ for all $A \in \mathcal{F}'$ and $\omega \in \Omega$. The existence of such a regular conditional distribution is well-known (see, for example, 1.1.6, 1.1.7 and 1.1.8 of [3]). Since τ' is a bounded stopping time, it follows from Doob's optional sampling theorem that

$$E^P(X_{\tau'}) = E^P(X_0).$$

Assume now that $E^P(X_\tau)$ is well-defined. Then $E^P(X_\tau) = E^P(E^P(X_\tau | F'))$, and the theorem can be proved by showing that

$$E^P(X_\tau | F') = X_{\tau'}, \quad \text{a.s.}[P].$$

Now the function $\omega \longrightarrow E^{Q_\omega}(X_\tau)$ is a version of $E^P(X_\tau | F')$ and so it will suffice to show

$$E^{Q_\omega}(X_\tau) = X_{\tau'}(\omega)$$

except for a set of ω's having P-probability zero. Notice that the existence of $E^P(X_\tau)$ implies there is a P-null set N_1 such that $E^{Q_\omega}(X_\tau)$ exists for $\omega \notin N_1$.

By Theorem 1.2.10 of [3], there exists another P-null set N_2 such that, for $\omega \notin N_2$, $\{X_t,\ t > \tau'(\omega)\}$ is an F_t-adapted martingale under Q_ω. Hence, for $\omega \notin N_2$, $\{X_t,\ t \geq 0\}$ is an F_t-adapted martingale under the probability measure $P_\omega = Q_\omega \circ T^{-1}_{\tau'(\omega)}$ where, for $s \geq 0$, T_s is the transformation on Ω defined by $(T_s\omega)(t) = \omega(s+t)$. For $\omega \notin N_1 \cup N_2$, let $p_{\tau'}(\omega)$ denote the restriction of ω to $[0,\tau'(\omega))$ and let $A_\omega = \{\omega' : \ \omega'(s) = \omega(s) \text{ for } 0 \leq s \leq \tau'(\omega)\}$. Because Q_ω is proper, $Q_\omega(A_\omega) = 1$. Furthermore, on the set A_ω,

$$X_\tau = X_{\tau[p_{\tau'}(\omega)]} \circ T_{\tau'(\omega)}.$$

Hence,

$$E^{Q_\omega}(X_\tau) = E^{P_\omega}(X_{\tau[p_{\tau'}(\omega)]}).$$

If $\tau'(\omega) < 1$, then $\tau(\omega) = \tau'(\omega)$ and $\tau[p_{\tau'}(\omega)] = 0$. If $\tau'(\omega) = 1$, then $p_{\tau'}(\omega)$ has length 1 and $\phi(\tau[p_{\tau'}(\omega)])$ is true. So, in either case,

$$\begin{aligned} E^{P_\omega}\left(X_{\tau[p_{\tau'}(\omega)]}\right) &= E^{P_\omega}(X_0) \\ &= E^{Q_\omega}(X_{\tau'}) \\ &= X_{\tau'}(\omega) \end{aligned}$$

The last equality uses the fact that Q_ω is proper.

The proof of the theorem is now complete.

Here is an example to show that condition (b) of the theorem can occur.

<u>Example</u>. Let $\{X_t\}$ be a standard Brownian motion process under P. Define

$$\tau(\omega) = 1 + e^{2X_1(\omega)^2}.$$

Then, given $X_1 = x$, $X_\tau - X_1$ is Gaussian with mean zero and variance e^{2x^2}. Hence,

$$\begin{aligned} E|X_\tau - X_1| &= E\left(E(|X_\tau - X_1| \; |X_1)\right) \\ &= \frac{2}{\sqrt{2\pi}} E\left(e^{X_1^2}\right) \\ &= \infty. \end{aligned}$$

Hence, $E|X_\tau| = \infty$. But X_τ is symmetrically distributed about 0. So $E(X_\tau)$ is undefined.

Remarks.

1. The proof above also works if Ω is any collection of right-continuous functions on $[0,\infty)$ such that (a) Ω is a complete, separable metric space and (b) whenever $\{p_n\}$ is a sequence of elements in Ω^*, each of length 1, such that $P_1 \ldots P_n \in \Omega^*$ for all n, ω defined by $\omega(s) = P_n(s-n+1)$, if $n-1 \le s < n$, $n \ge 1$, belongs to Ω.

An example of such an Ω, besides $C[0,\infty)$ and $D[0,\infty)$, is the collection of all right continuous functions on $[0,\infty)$ which are constant on intervals of the form $[n-1,n)$ where n is a positive integer.

2. A discrete-time version of our theorem holds on $\Omega = \mathbb{R}^\infty$, the countably infinite product of the real line, for nonnegative integer valued stop rules. This could be proved the same way by using discrete-time analogues of the induction lemma and Theorem 1.2.10 of [3]. We could alternatively obtain it as a corollary of our theorem in the continuous case by identifying $\mathbb{R}^\infty$ with the collection of all right continuous functions on $[0,\infty)$ which are constant on intervals of the form $[n-1,n)$, where n is a positive integer.

3. It is possible to obtain in an obvious way a version of our theorem where the random variables forming the martingale are not necessarily coordinate random variables. Such a version would be proved by reducing it to the coordinate variables case by a change of variable.

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